数理解析研究所講究録 1459 巻 2005 年 57-63

AW^*-triples, partial $*$-isomorphisms and Morita equivalence

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Let B and C be two AW^*-subalgebras of an AW^*-algebra A. In this talk, we describe the relative position of B and C in A (e.g., Morita equivalence, etc.) in terms of AW^*-subtriples of A, the normalizer of B and C in A, or partial $*$-isomorphisms between B and C (the definitions will be given below). Then, under a stronger assumption of B and C being monotone complete, we show that the set of these AW^*-subtriples is embedded naturally in an inverse semigroup associated with B and C. The key idea is to regard AW^*-algebras and AW^*-triples as a generalization of projections and partial isometries.

1 Introduction

If X is a linear subspace of a $*$-algebra A, the two conditions that $X^2 \subset X = X^*$ (X being a $*$-subalgebra of A) and that $XX^*X \subset X$ (we call such an X a **subtriple** of A) are regarded as a generalization of the notions of projection ($p^2 = p = p^*$) and partial isometry ($xx^*x = x$), respectively. Here, for $X, Y, Z \subset A$, we write $XY := \{xy : x \in X, y \in Y\}$, $X^2 := XX$, $X^* := \{x^* : x \in X\}$, and $XYZ := (XY)Z = X(YZ)$. For a subtriple X of A the sets $B := \operatorname{lin} XX^*$ and $C := \operatorname{lin} X^*X$ (lin denotes linear span) are $*$-subalgebras of A, and the relation among B, C and X is viewed as an analogue of the relation among Murray-von Neumann equivalent projections and the partial isometry implementing the equivalence.

*From October 1, 2005 or later the author's affiliation and the English name of the university have changed due to the merger of three universities.

If we adjust the above situation slightly, we obtain the notions of strong Morita equivalence for C^*-algaberas and Morita equivalence for von Neumann algebras in the sense of M. Rieffel [7]. That is, two C^*-algebra B and C are strong Morita equivalent if there exist a C^*-algebra A containing B and C as C^*-subalgebras and a norm closed subtriple X of A such that $B = \overline{\mathrm{lin}}\, XX^*$ and $C = \overline{\mathrm{lin}}\, X^*X$. Two von Neumann algebras B and C are Morita equivalent if there exist a von Neumann algebra A containing B and C as von Neumann subalgebras and a σ-weakly closed subtriple X of A such that $B = \overline{\mathrm{lin}}^{\sigma}\, XX^*$ and $C = \overline{\mathrm{lin}}^{\sigma}\, X^*X$. Here $\overline{\{\}}$, $\overline{\{\}}^{\sigma}$ denote respectively norm closure and σ-weak closure. The linking algebra technique ([3]) shows that the definitions of (strong) Morita equivalences above are equivalent to the usual ones, which are defined in terms of imprimitivity bimodules.

Subtriples in the above sense arise naturally in the theory of operator algebras. The following fact, which will be worked out in another paper, was a motivation for considering them and introducing an inverse semigroup structure of them in [4]. Let A be a von Neumann algebra and let $\{X_g\}_{g\in G}$ be a family of σ-weakly closed linear subspaces of A indexed by a discrete group G such that

$$X_g^* = X_{g^{-1}}, \quad X_{g_1}X_{g_2} \subset X_{g_1g_2}, \quad \forall g,\, g_1,\, g_2 \in G$$

(such a family corresponds to each coaction of G on A). Then each X_g is a subtriple of A, the algebraic direct sum $\mathcal{A} := \bigoplus_{g\in G} X_g$, with the product and involution inherited from A, is a G-graded $*$-algebra, and under a certain technical assumption, $\mathcal{A}$ (and A also if $\{X_g\}$ is associated with a coaction of G) is viewed as the twisted crossed product $B \rtimes_{\theta,u} G$ with respect to a twisted action $(\theta,\, u)$ of G on the von Neumann subalgebra $B := X_e$, and is described in terms of only B and G. Indeed, it follows from Theorem 1 below that each $X_g = Bs_gB$ for some $s_g \in \mathrm{PI}\, A$ (partial isometries of A). If B is σ-finite and properly infinite, then we may take s_g so that $X_g = Bs_g = s_gB$ and $s_g^* = s_{g^{-1}}$, and $\theta : G \to \mathrm{PAut}\, B$ (the set of all partial $*$-automorphims of B, i.e., $*$-isomorphisms between reduced subalgebras of B) and $u : G \times G \to \mathrm{PI}\, B$ are defined by $\theta_g := \mathrm{Ad}\, s_g : s_g^*s_gB \to s_gs_g^*B$, $x \mapsto s_gxs_g^*$, and $u(g_1,\, g_2) := s_{g_1}s_{g_2}(s_{g_1g_2})^*$ so that $\theta_{g_1} \circ \theta_{g_2} = \mathrm{Ad}\, u(g_1,\, g_2) \circ \theta_{g_1g_2}$, u satisfies the 2-cocycle condition, and the product and involution in $\mathcal{A}$ are given in terms of $(\theta,\, u)$.

The work in this talk was intended to generalize, and simplify the proofs of, part of the results in [4].

2 Invertible bimodules and normalizers

In this section A denotes a fixed AW^*-algebra ([5], [1]), and $\mathcal{S}(A)$ denotes the set of all AW^*-subalgebras of A.

Definition (Invertible bimodule, MR-equivalence in an AW^*-algebra). (i) For $B, C \in \mathcal{S}(A)$ we call $X \subset A$ an **invertible B-C-bimodule in A** if

$$L(X) := \begin{bmatrix} B & X \\ X^* & C \end{bmatrix} \subset M_2(A) \text{ is an } AW^*\text{-subalgebra of } M_2(A).$$

Here $M_2(A)$ (the algebra of 2×2 matrices over A) is an AW^*-algebra ([2]). Then $BX + XC \subset X$, $XX^* \subset B$, $X^*X \subset C$; hence

(1) X is both a sub-B-C-bimodule and a subtriple of A,
$\exists$ left inner product $\langle \cdot, \cdot \rangle_l : X \times X \to B$, $(x, y) \mapsto xy^*$,
$\exists$ right inner product $\langle \cdot, \cdot \rangle_r : X \times X \to C$, $(x, y) \mapsto x^*y$,
$\exists$ triple product $[\cdot, \cdot, \cdot] : X \times X \times X \to X$, $[x, y, z] := xy^*z$;

(2) $\exists\, h \in \operatorname{Proj} Z(B)$ (resp. $\exists\, k \in \operatorname{Proj} Z(C)$): $M_l(X) := M(K_l(X)) = hB$, $M_r(X) := M(K_r(X)) = kC$, where $\operatorname{Proj}(\cdot)$ denotes the set of projections, $Z(\cdot)$ denotes the center, $K_l(X) := \overline{\operatorname{lin}}\, XX^*$ (resp. $K_r(X) := \overline{\operatorname{lin}}\, XX^*$) is a norm closed two-sided ideal of B (resp. C), and $M(\cdot)$ denotes the multiplier algebra of a C^*-algebra.

We write $\mathrm{INV}_A(B, C)$ for the set of all invertible B-C-bimodules in A.

(ii) We call $B, C \in \mathcal{S}(A)$ **MR (Morita-Rieffel) equivalent in** A and write $B \sim_A C$ if $\exists\, X \in \mathrm{INV}_A(B, C)$: $B = M_l(X)$, $C = M_r(X)$.

Definition (Normalizer). For $B, C \in \mathcal{S}(A)$ we call the following sets the **normalizer** and the **regular normalizer** of B, C in A, respectively ($\mathrm{PI}\, A$ denotes the set of all partial isometries in A):

$$\begin{aligned} N_A(B, C) &:= \{x \in A : xCx^* \subset B,\ x^*Bx \subset C,\ xx^* \in B,\ x^*x \in C\}, \\ RN_A(B, C) &:= \{s \in \mathrm{PI}\, A \cap N_A(B, C) : \exists\, h \in \operatorname{Proj} Z(B), \exists\, k \in \operatorname{Proj} Z(C) : \\ &\qquad h \le ss^*, k \le s^*s,\ s = hs + sk\}. \end{aligned}$$

Theorem 1. Let A be an AW^*-algebra and $B, C \in \mathcal{S}(A)$.

(i) For $X \subset A$, $X \in \mathrm{INV}_A(B, C) \iff \exists\, s \in RN_A(B, C)$: $X = BsC$.
In this case, $M_l(X) = C_B(ss^*)B$, $M_r(X) = C_C(s^*s)C$ ($C_B(\cdot)$ and $C_C(\cdot)$ denote the the central cover of a projection in B and in C, respectively);

$$\mathrm{PI}\, X := X \cap \mathrm{PI}\, A = \{usv : u \in \mathrm{PI}\, B,\ v \in \mathrm{PI}\, C,\ u^*u \le ss^*,\ vv^* \le s^*s,\ u^*u = svv^*s^*\}.$$

(ii) For $s, t \in RN_A(B, C)$, $BsC = BtC \iff \exists\, u \in \mathrm{PI}\, B$, $v \in \mathrm{PI}\, C$: $t = usv$, $u^*u = ss^*$, $s^*s = vv^*$.

(iii) $B \sim_A C \iff \exists\, s \in RN_A(B, C)$: $C_B(ss^*) = 1_B$, $C_C(s^*s) = 1_C$.

(iv) $N_A(B, C) = B \cdot RN_A(B, C) \cdot C = \bigcup\{X : X \in \mathrm{INV}_A(B, C)\}$.

3 Invertible bimodules and partial *-isomorphisms

Definition ((Abstract) invertible bimodule). (i) Let B and C be AW^*-algebras. We call a linear space X an **invertible** B-C-**bimodule** if it is a B-C-bimodule and there exist maps $\langle\cdot,\,\cdot\rangle_l : X\times X\to B$, $\langle\cdot,\,\cdot\rangle_r : X\times X\to C$ such that $L(X) := \begin{bmatrix} B & X \\ X^* & C\end{bmatrix}$ is an AW^*-algebra with the following product and involution:

$$\begin{bmatrix} b_1 & x_1 \\ y_1^* & c_1\end{bmatrix}\begin{bmatrix} b_2 & x_2 \\ y_2^* & c_2\end{bmatrix} = \begin{bmatrix} b_1b_2 + \langle x_1,\, y_2\rangle_l & b_1x_2 + x_1c_2 \\ y_1^*b_2 + c_1y_2^* & \langle y_1,\, x_2\rangle_r + c_1c_2\end{bmatrix},\quad \begin{bmatrix} b_1 & x_1 \\ y_1^* & c_1\end{bmatrix}^* = \begin{bmatrix} b_1^* & y_1 \\ x_1^* & c_1^*\end{bmatrix}.$$

Here X^* denotes the set of all x^*, $x\in X$, which is made into a C-B-bimodule by the following operations:

$$\lambda x^* = (\bar{\lambda}x)^*,\quad cx^*b = (b^*xc^*)^*\quad (\lambda\in\mathbb{C},\, b\in B,\, c\in C,\, x\in X).$$

Then it follows that $\langle\cdot,\,\cdot\rangle_l$ and $\langle\cdot,\,\cdot\rangle_r$ satisfy the usual properties of inner products.

(ii) We call two AW^*-algebras B and C **MR (Morita-Rieffel) equivalent** and write $B\sim C$ if $\exists$ invertible B-C-bimodule X: $M_l(X) = B$, $M_r(X) = C$.

We write $\mathrm{INV}(B,\, C)$ for the set of all invertible B-C-bimodules. If, in particular, $B = C$, we abbreviate this to $\mathrm{INV}(B) := \mathrm{INV}(B,\, B)$, and call its element an **invertible** B-**bimodule**.

(iii) We call a map $\tau : X\to Y$ between $X,\, Y\in \mathrm{INV}(B,\, C)$ a **module monomorphism** if it is a B-C-bimodule map and preserves the inner products (i.e., $\tau(bxc) = b\tau(x)c$, $\langle\tau(x),\, \tau(y)\rangle_l = \langle x,\, y\rangle_l$, $\langle\tau(x),\, \tau(y)\rangle_r = \langle x,\, y\rangle_r$, $\forall\, x,\, y\in X$, $b\in B$, $c\in C$). A surjective module monomorphism is called a **module isomorphism**.

We call $X,\, Y\in\mathrm{INV}(B,\, C)$ **isomorphic** and write $X\cong Y$ if $\exists$ a module isomorphism $X\to Y$.

$X\in\mathrm{INV}(B,\, C) \Rightarrow X^*\in\mathrm{INV}(C,\, B)$, $(X^*)^* = X$. Here we define the inner products in X^* by $\langle x^*,\, y^*\rangle_l := \langle x,\, y\rangle_r\in C$, $\langle x^*,\, y^*\rangle_r := \langle x,\, y\rangle_l\in B$ $(x,\, y\in X)$.

Definition (Partial *-isomorphism). By a **partial *-isomorphism** between AW^*-algebras C and B we mean a *-isomorphism of the form $\theta : r(\theta)Cr(\theta)\to l(\theta)Bl(\theta)$, where $r(\theta)\in\mathrm{Proj}\, C$ and $l(\theta)\in\mathrm{Proj}\, B$. We call the partial *-isomorphism θ **positive** (resp. **negative**) if $r(\theta)\in\mathrm{Proj}\, Z(C)$ (resp. $l(\theta)\in\mathrm{Proj}\, Z(B)$); **central** if it is both positive and negative; and **regular** if $\exists$ positive θ_1, $\exists$ negative θ_2: $\theta = \theta_1\oplus\theta_2$. Here, when two partial *-isomorphisms θ_i, $i = 1,\, 2$, satisfy the condition $C_C(r(\theta_1))C_C(r(\theta_2)) = 0 = C_B(l(\theta_1))C_B(l(\theta_2))$ (hence $(r(\theta_1) + r(\theta_2))C(r(\theta_1) + r(\theta_2)) = r(\theta_1)Cr(\theta_2) + r(\theta_2)Cr(\theta_2)$, and similarly for $l(\cdot)$), a partial *-isomorphism $\theta_1 \oplus \theta_2$ is defined by

$$r(\theta_1 \oplus \theta_2) := r(\theta_1) + r(\theta_2), \quad l(\theta_1 \oplus \theta_2) := l(\theta_1) + l(\theta_2),$$
$$(\theta_1 \oplus \theta_2)(x_1 + x_2) := \theta_1(x_1) + \theta_2(x_2), \quad x_i \in r(\theta_i)Cr(\theta_i).$$

We write $\mathrm{PIsom}(B, C)$ for the set of all partial *-isomorphisms between C and B, and $\mathrm{PIsom}(B, C)^+$, $\mathrm{PIsom}(B, C)^-$, $\mathrm{PIsom}(B, C)^0$, and $\mathrm{RPIsom}(B, C)$ for the sets of all positive, negative, central, and regular ones, respectively.

Definition (Invertible bimodule associated with a regular partial *-isomorphism). For $\theta = \theta_1 \oplus \theta_2 \in \mathrm{RPIsom}(B, C)$ with θ_1 positive and θ_2 negative we define $\langle\theta\rangle \in \mathrm{INV}(B, C)$ as the set $Bl(\theta_1) \oplus r(\theta_2)C$ with the following module operation, inner products, and triple product:
$\forall b \in B, \forall c \in C, \forall x_1, y_1, z_1 \in Bl(\theta_1), \forall x_2, y_2, z_2 \in r(\theta_2)C$:

$$\begin{aligned} b \cdot (x_1 \oplus x_2) \cdot c &:= bx_1\theta_1(r(\theta_1)c) \oplus \theta_2^{-1}(l(\theta_2)b)x_2c, \\ \langle x_1 \oplus x_2, y_1 \oplus y_2\rangle_l &:= x_1y_1^* + \theta_2(x_2y_2^*) \in B, \\ \langle x_1 \oplus x_2, y_1 \oplus y_2\rangle_r &:= \theta_1^{-1}(x_1^*y_1) + x_2^*y_2 \in C, \\ [x_1 \oplus x_2, y_1 \oplus y_2, z_1 \oplus z_2] &:= x_1y_1^*z_1 \oplus x_2y_2^*z_2 = \langle x_1 \oplus x_2, y_1 \oplus y_2\rangle_l \cdot (z_1 \oplus z_2) \\ &= (x_1 \oplus x_2) \cdot \langle y_1 \oplus y_2, z_1 \oplus z_2\rangle_r. \end{aligned}$$

Definition (Equivalence for partial *-isomorphisms). Define $\theta, \psi \in \mathrm{PIsom}(B, C)$ to be equivalent, $\theta \sim \psi$, if $\exists u \in \mathrm{PI}\,B$, $\exists v \in \mathrm{PI}\,C$: $v^*v \le r(\psi) \le C_C(v^*v)$, $vv^* \le r(\theta) \le C_C(vv^*)$, $\theta(vv^*) = u^*u$, $\psi|v^*vCv^*v = (\mathrm{Ad}\,u) \circ \theta \circ (\mathrm{Ad}\,v)$.
We denote by $[\theta]$ the equivalence class in $\mathrm{PIsom}(B, C)$ containing θ, and by $[S]$ the set of all equivalence classes containing elements of $S \subset \mathrm{PIsom}(B, C)$.

Proposition 2. If B and C are AW^*-algebras, then $\forall X \in \mathrm{INV}(B, C)$, $\exists \theta \in \mathrm{RPIsom}(B, C)$: $X \cong \langle\theta\rangle$, and hence $\exists$ bijection $[\mathrm{INV}(B, C)] \longleftrightarrow [\mathrm{RPIsom}(B, C)]$, $[BsC] \longleftrightarrow [\mathrm{Ad}\, s]$.

Theorem 3. Let B and C be monotone complete C^*-algebras (and hence AW^*-algebras; here a C^*-algebra is called **monotone complete** if every bounded increasing net in its self-adjoint part has a supremum).

(i) The following conditions are equivalent:
(1) $B \sim C$ (MR-equivalent);
(2) $\exists \theta \in \mathrm{RPIsom}(B, C)$: $C_B(l(\theta)) = 1_B$, $C_C(r(\theta)) = 1_C$;
(3) $\exists \theta \in \mathrm{PIsom}(B, C)$: as in (2).

(ii) $\exists$ monotone complete C^*-algebra D containing B and C as monotone closed C^*-subalgebras: each element of $\mathrm{INV}(B, C)$ or $\mathrm{PIsom}(B, C)$ is realized via a partial isometry of D, i.e., $\forall X \in \mathrm{INV}(B, C)$, $\exists s \in RN_D(B, C)$ (the normalizer of B, C in D): $X \cong BsC$, $\forall \theta \in \mathrm{PIsom}(B, C)$, $\exists s \in RN_D(B, C)$; $\theta \sim \mathrm{Ad}\, s$.

(iii) $\exists$ inverse semigroup S (cf., e.g., [6]): $[\mathrm{INV}(B, C)] \cong [\mathrm{PIsom}(B, C)]$ is a subtriple of S. Here $T \subset S$ is called a subtriple of S if $TT^{-1}T = T$ (and so the triple product $[x, y, z] := xy^{-1}z$ is defined in T). Moreover the triple products in $[\mathrm{INV}(B, C)]$ and in $[\mathrm{PIsom}(B, C)]$ are described in terms of the tensor product of bimodules and the composition of maps, respectively.

4 AW^*-triples

Definition (AW^*-triple). (i) Let A be an AW^*-algebra. We call $X \subset A$ an **AW^*-subtriple** of A if $\exists\, B, C \in \mathcal{S}(A)$: $X \in \mathrm{INV}_A(B, C)$.

(ii) By an **AW^*-triple** we mean an AW^*-subtriple of some AW^*-algebra. Here we identify two AW^*-triples X and Y if $\exists$ **triple isomorphism** $\tau : X \to Y$ (a linear bijection satisfying the condition

$$\tau([x, y, z]) = [\tau(x), \tau(y), \tau(z)], \ \forall x, y, z \in X),$$

i.e., we consider only the triple products forgetting the bimodule structures.

Proposition 4. (i) Every AW^*-triple X is written in the following form:

$$X = X^{++} \oplus X^0 \oplus X^{--}, \quad X^{++} \cong A_1 e, \ \ X^0 \cong A_2, \ \ X^{--} \cong fA_3,$$

where A_i, $i = 1, 2, 3$, are AW^*-algebras, $e \in \mathrm{Proj}\, A_1$, $C_{A_1}(e) = 1_{A_1}$,

$$\exists\, h \in \mathrm{Proj}\, Z(A_1),\ \exists\, u \in \mathrm{PI}\, A_1 : \ \ he = uu^*,\ u^*u = h \ \Rightarrow\ h = 0;$$

$f \in \mathrm{Proj}\, A_3$, $C_{A_3}(f) = 1_{A_3}$, f satisfies the condition similar to the above; and the triple products in $A_1 e$, A_2, fA_3 are given by $[x, y, z] := xy^*z$.

(ii) For AW^*-triples X, Y, write $X^{++} = A_1 e$, $X^0 = A_2$, $X^{--} = fA_3$, $Y^{++} = B_1 p$, $Y^0 = B_2$, $Y^{--} = qB_3$ as above. Then $\tau : X \to Y$ is a triple isomorphism $\iff$ $\tau(X^{++}) = Y^{++}$, $\tau(X^0) = Y^0$, $\tau(X^{--}) = Y^{--}$, $\exists$ *-isomorphisms $\alpha : A_1 \to B_1$, $\beta : A_2 \to B_2$, $\gamma : A_3 \to B_3$, $\exists\, u \in \mathrm{PI}\, B_1$, $\alpha(e) = uu^*$, $u^*u = p$, $\exists\, v \in B_2$: unitary, $\exists\, w \in \mathrm{PI}\, B_3$, $\gamma(f) = w^*w$, $ww^* = q$:

$$\tau | X^{++} = \alpha(\cdot)u, \ \ \tau | X^0 = \beta(\cdot)v, \ \ \tau | X^{--} = w\gamma(\cdot).$$

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